VIBRATION OF PRESSURIZED THIN-WALLED CYLINDER INDUCED BY PULSED LASER

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Abstract

Differential equations are presented that describe wall oscillations of a gas-pressurized thin-walled cylinder for cases where variations in the axial direction are negligible. A linear stress/strain relation is assumed. Harmonic solutions are obtained. Limiting forms of these solutions are given for cases where the mode number, n = 0, 1, 2,…, is moderate and for cases where the mode number is large. In the former case, curvature effects are important. In the latter case, the vibrations behave locally like those on a flat plate. The applicability of the harmonic solutions, for evaluating the hoop stress induced by a high energy pulsed laser beam, is discussed. The maximum stress induced during the initial transient is investigated. An estimate is given for the maximum stress perturbation induced by a "slab" type (no axial variation) beam with uniform fluence and a width equal to the cylinder diameter. In this case, the estimate for the maximum value of the laser induced hoop stress perturbation, σ_m, is found from $\sigma_m/\sigma_i = (CF/pa)(E'/\rho)^{1/2}$ where, σ_i, C, F, p, a, E', and ρ are initial pressure induced hoop stress, coupling coefficient, fluence, cylinder internal pressure, cylinder radius, effective wall modulus of elasticity, and wall density, respectively. The maximum stress perturbation induced by slab and circular cross-section laser beams, with non-uniform profiles and widths which are small compared with cylinder radius, are also estimated. For the case of a narrow slab beam, the upper bound on the laser induced hoop stress is found from $\sigma_m/\sigma_i = 0.5(CF_0/pa)^2\,(E'/\rho)$ where F_0 is center line fluence.

1. Introduction

The interaction between a high-energy laser pulse and a surface can result in material blow off and impulse loading. The latter may induce vibration and/or structural damage. As a result, high-energy pulse lasers are often considered as a component of a missile defense system.[1]

Research in the area of pulsed-laser surface interaction has focused on the determination of the laser levels needed to induce blow-off and determination of the

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coupling coefficient that relates incident fluence to impulse.[1] The subsequent effect on the structure has received less attention. In this connection, Sutton[2] has evaluated the wall vibration induced in a pressurized cylinder by a high-energy laser pulse of uniform width. However, Sutton's model neglects curvature effects, which, in fact, play a significant role in the structural response for beam widths of the order of cylinder diameter. Hence, a study of the vibrational response of a gas pressurized cylinder, to a high-energy laser pulse, including curvature effects, was undertaken. A linear stress/strain relation is assumed. Emphasis is placed on the case treated by Sutton[2], namely, an incident laser beam of uniform fluence with width equal to the cylinder diameter and infinite axial extent. (Beams of infinite axial extent, with a fluence profile that does not vary with axial position, are referred to as "slab" beams, herein). Slab and circular cross-section laser beams, with widths that are small compared to the cylinder diameter, are also considered.

The response of the cylinder is expected to consist of an initial transient followed by harmonic motion. In Section 2 (Analysis), equations are presented which describe the cylinder wall motion in the limit of negligible end wall effects. Harmonic solutions are obtained. In Section 3 (Applications), the applicability of these harmonic solutions, for evaluation of laser induced vibration, is discussed. The initial transient, caused by the laser pulse, is also discussed. Estimates are given for the initial maximum hoop stress perturbation induced by the laser pulse. A comparison of the present results with those of Sutton is given in Section 4. The effect of cylinder translation is discussed in Section 5.

2. Analysis

We consider wall vibrations of a gas pressurized cylinder of radius a, axial length L, thickness h, internal pressure p, and wall density ρ (Fig. 1) and assume $h/a \ll 1$ and $a/L \ll 1$.

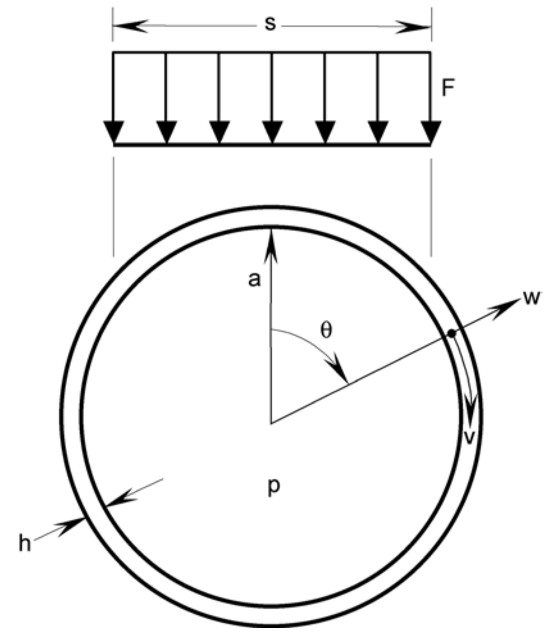

Figure 1. Notation for evaluation of wall vibrations induced in pressurized cylinder due to pulsed laser beam.

The initial pressure-induced hoop stress σ_i is

$$\sigma_i = \frac{pa}{h} \tag{1}$$

Let w and v denote small displacements in the radial and tangential directions, respectively, of the center of an arc element. These are functions of the initial angular position θ and time t, and define the displacement of the centroid of each arc element from its initial (t = 0) position. Variations of w and v in the axial direction are ignored. This is consistent with the neglect of end wall effects (i.e., the assumption $a/L \ll 1$) and the assumption of a slab beam with a fluence profile that does not vary in the axial direction. The neglect of axial derivatives in Eqs. 4 of Fung[3] leads to momentum conservation equations given by

$$\frac{1}{\omega_o^2}\frac{\partial^2 w}{\partial t^2} = \varepsilon_i\left(\frac{\partial^2 w}{\partial \theta^2} - \frac{\partial v}{\partial \theta}\right) +$$
$$\frac{1}{12}\left(\frac{h}{a}\right)^2\left(\frac{\partial^3 v}{\partial \theta^3} - \frac{\partial^4 w}{\partial \theta^4}\right) - \left(\frac{\partial v}{\partial \theta} + w\right) \tag{2a}$$

$$\frac{1}{\omega_o^2}\frac{\partial^2 v}{\partial t^2} = \frac{\partial w}{\partial \theta} + \frac{\partial^2 v}{\partial \theta^2} + \frac{1}{12}\left(\frac{h}{a}\right)^2\left(\frac{\partial^2 v}{\partial \theta^2} - \frac{\partial^3 w}{\partial \theta^3}\right) \tag{2b}$$

where

$$\omega_o^2 = \frac{E}{1-\nu^2}\frac{1}{\rho a^2} = \frac{E'}{\rho a^2} \tag{2c}$$

$$\varepsilon_i = \frac{1-\nu^2}{E}\sigma_i = \frac{\sigma_i}{E'} \tag{2d}$$

Here, ν, E, and $E' = E/(1-\nu^2)$ are Poisson's ratio, the modulus of elasticity and an "effective" modulus of elasticity, respectively. The quantity ω_0 is the fundamental cylinder vibration frequency and ε_i is a parameter which characterizes the initial radial strain induced by pressurization. For cases where the separation distance between the end walls is kept constant during the pressurization process, the axial strain is zero and ε_i is the actual radial strain, w_i/a. If the end walls are unconstrained during pressurization, an axial stress equal to half the hoop stress is induced, and the corresponding radial strain is then $w_i/a = 0.5(2-\nu)\varepsilon_i/(1-\nu^2)$.

Harmonic solutions of Eqs. 2 are now obtained. Assume perturbations of the form

$$w = \sum w_n \cos n\theta \sin \omega_n t \tag{3a}$$

$$v = \Sigma\, v_n \sin n\theta \sin \omega_n t \tag{3b}$$

where n is the vibrational mode number which, for each mode, equals the number of sine waves around the cylinder circumference. Substitution of Eqs. 3 into Eqs. 2 yields

$$\frac{2\left(\omega_n/\omega_o\right)^2}{k_3 + n^2 k_4} = 1 \pm \left[1 - \frac{4n^2\left(k_2 k_3 - k_1 k_4\right)}{\left(k_3 + n^2 k_2\right)^2}\right]^{1/2} \tag{4a}$$

$$\frac{v_n}{w_n} = \frac{n k_4}{\left(\omega_n/\omega_o\right)^2 - n^2 k_2} \tag{4b}$$

where

$$k_1 = 1 + \varepsilon_i + \frac{1}{12}\left(\frac{h}{a}\right)^2 n^2 \tag{5a}$$

$$k_2 = 1 + \frac{1}{12}\left(\frac{h}{a}\right)^2 \tag{5b}$$

$$k_3 = 1 + \varepsilon_i n^2 + \frac{1}{12}\left(\frac{h}{a}\right)^2 n^4 \tag{5c}$$

$$k_4 = 1 + \frac{1}{12}\left(\frac{h}{a}\right)^2 n^2 \tag{5d}$$

The kinetic energy in each mode, at t = 0, is

$$\frac{2\left(KE\right)_n}{\pi\rho h a \omega_n^2} = w_n^2\left[1 + \left(\frac{v_n}{w_n}\right)^2\right] \tag{6}$$

Equation 6 defines the net (kinetic plus potential) energy associated with the vibrational motion. Let σ and ε denote the perturbation in the tensile (hoop) stress and in strain. These are related by

$$\sigma = E'\varepsilon \tag{7}$$

An expression for ε, in terms of v and w, is given by Eq. A-2, namely

$$\varepsilon = \frac{w}{a} + \frac{1}{a}\frac{\partial v}{\partial \theta} + \frac{1}{2a^2}\left[\left(\frac{\partial w}{\partial \theta}\right)^2 + \left(\frac{\partial v}{\partial \theta}\right)^2\right] \tag{8}$$

Further discussion relating to the computation of strain is given in Appendix A.

Simplified equations for the frequency, velocity ratio, and strain associated with the positive and negative roots of Eq. 4a are now given. We make the realistic assumption

$$\varepsilon_i << 1, \qquad \frac{1+n^2}{12}\left(\frac{h}{a}\right)^2 << 1 \tag{9}$$

If terms which are small, compared with one, are neglected, the dependent variables associated with the positive root of Eq. 4a become

$$\left(\omega_n^+/\omega_o\right)^2 = 1+n^2 \tag{10a}$$

$$v_n^+/w_n^+ = n \tag{10b}$$

$$\varepsilon_n^+ = \left(1+n^2\right)\left(w_n^+/a\right)\cos n\theta \sin \omega_n^+ t + \frac{1}{2}\left[\frac{nw_n^+}{a}\right]^2\left[\sin^2 n\theta + n^2 \cos^2 n\theta\right]\sin^2 \omega_n^+ t \tag{10c}$$

Similarly, the dependent variables associated with the negative root of Eq 4a become

$$\left(\omega_n^-/\omega_o\right)^2 = \frac{n^2\left(n^2-1\right)\left[\varepsilon_i + \frac{n^2-1}{12}\left(\frac{h}{a}\right)^2\right]}{1+n^2} \tag{11a}$$

$$v_n^-/w_n^- = -1/n \tag{11b}$$

$$\varepsilon_n^- = \frac{1}{2}\left(\frac{nw_n^-}{a}\right)^2 \sin^2 n\theta \sin^2 \omega_n^- t \tag{11c}$$

Note that $\omega_n^- = 0$ for $n = 0,1$.

It can now be shown that the positive and negative roots of Eq.4a have the following physical significance. Substitution of Eq. 10b into Eq. 2b indicates that ω_n^+ is a consequence of the conservation of v (transverse) momentum. Similarly, substitution of Eq. 11b into Eq. 2a indicates that ω_n^- is a consequence of the conservation of w (radial) momentum.

In the limit of large n, namely 1/n << 1, Eqs. 11 become

$$\left(\omega_n^-/\omega_o\right)^2 = n^2\left[\varepsilon_i + \frac{n^2}{12}\left(\frac{h}{a}\right)^2\right] \tag{12a}$$

$$v_n^-/w_n^- << 1 \tag{12b}$$

$$\varepsilon_n^- = \frac{1}{2}\left(\frac{nw_n^-}{a}\right)^2 \sin^2 n\theta \sin^2 \omega_n^- t \tag{12c}$$

These are the equations that describe the fundamental vibration mode for a flat plate of width s = πa/n. This is a consequence of the fact that, for large n, the arc length between radial velocity nodes, namely πa/n, is small compared to the cylinder radius a and thus departs only slightly from a flat plate.

Equations 10c, 11c, and 12c define the strain associated with a single mode. In the case of multiple modes, a summation process is needed as discussed in Appendix A.

3. Applications

We now investigate the stress perturbation induced by an energetic laser pulse of width s, incident on the cylinder surface. It is assumed that ablation creates an impulsive load which results in inward momentum of the cylinder surface. The resulting stress perturbation is estimated.

Consider the case of a laser pulse of uniform fluence F and width s (Fig. 1) incident on the cylinder surface at time $t = 0^+$. Let C represent the coupling coefficient, namely the ratio of the induced impulse (radial momentum/area) to the incident fluence (energy/area). The initial conditions regarding cylinder surface motion are then, for $t = 0^+$,

$$w = v = \partial v/\partial t = 0 \tag{13a}$$

$$\frac{\partial w}{\partial t} = (-1)\frac{CF}{\rho h}\cos\theta \qquad |\theta| \le \theta_s \tag{13b}$$

$$= 0 \qquad \theta_s < |\theta| \le \pi \tag{13c}$$

where θ_s is the value of θ corresponding to the edge of the laser beam,

$$\theta_s = \sin^{-1}\left[s/(2a)\right] \tag{13d}$$

The initial condition $v = \partial v/\partial t = 0$ is clearly inconsistent with Eqs. 3 and 4b so that those equations cannot be used directly to construct a solution which satisfies the initial condition v = 0, except for the case of large n. (See Eq. 12b). There will be an initial transient in the region $|\theta| \le \theta_s$ which later spreads to encompass the entire cylinder and evolves into a multimode harmonic motion. It is expected that the peak induced stress will occur during the initial transient [times of order $\omega t = 0(1)$]. An estimate of this peak stress is given below for the cases s = 2a and s/a << 1.

Case s = 2a:

In view of the initial condition v = 0, an estimate of the peak stress, during the initial transient, can be found from Eq. 2a by assuming v = 0, therein, and by assuming that the vibration remains confined to the region $|\theta| \leq \pi/2$. Thus we assume cylinder wall perturbations of the form

$$w = (-1)\frac{CF}{\rho h\omega}\cos\theta\sin\omega t \quad (14a)$$

$$v = 0 \quad (14b)$$

$$\varepsilon = w/a \quad (14c)$$

for $|\theta| \leq \pi/2$. Substitution into Eq. 2a, and neglect of terms of order ε_i and $(h/a)^2/12$ yields

$$\omega^2 = \omega_0^2 \quad (15)$$

The maximum strain ε_m occurs at $\theta = 0$ and $\omega t = 3\pi/2$, and equals

$$\varepsilon_m = \frac{CF}{\rho h}\left(\frac{\rho}{E'}\right)^{1/2} \quad (16a)$$

which may be viewed as providing an upper limit on the induced strain. The maximum hoop stress, due to the laser pulse, is found from $\sigma_m = E'\varepsilon_m$, or

$$\frac{\sigma_m}{\sigma_i} = \frac{CF}{pa}\left(\frac{E'}{\rho}\right)^{1/2} \quad (16b)$$

An estimate of the stress perturbation associated with the long term harmonic motion is now found. We assume that Eqs. 3 to 10 apply and that the induced vibration is primarily the n = 1 mode with energy equal to the kinetic energy deposited at t = 0. This choice most closely matches the initial radial velocity profile for the region $|\theta| \leq \pi/2$, and corresponds to the use of the quantities w_1^+ and ω_1^+. The superscript plus sign is henceforth omitted. The initial kinetic energy is

$$\frac{KE}{\pi\rho ha} = \frac{1}{4}\left(\frac{CF}{\rho h}\right)^2 \quad (17)$$

The energy in mode n = 1 is (Eq. 6)

$$\frac{(KE)_1}{\pi\rho ha} = 2\omega_0^2 w_1^2 \quad (18)$$

Equating Eqs. 17 and 18, and recalling Eq. 10, indicates

$$\frac{w_1}{a} = \frac{CF}{\rho h}\left(\frac{1}{8}\,\frac{\rho}{E'}\right)^{1/2} \quad (19a)$$

$$\omega_1^2 = 2\omega_0^2 \quad (19b)$$

The maximum strain and stress are given by

$$\varepsilon_m = \frac{1}{\sqrt{2}}\frac{CF}{\rho h}\left(\frac{\rho}{E'}\right)^{1/2} \quad (20a)$$

$$\frac{\sigma_m}{\sigma_i} = \frac{1}{\sqrt{2}}\frac{CF}{pa}\left(\frac{E'}{\rho}\right)^{1/2} \quad (20b)$$

Equation 20b is only a factor $1/\sqrt{2}$ less than the value in Eq. 16b. The degree to which Eq. 20b characterizes the late time oscillation requires further study.

Case s/a << 1:

In the present case, the laser spot size is small, relative to the cylinder radius, and a flat plate approximation can be made (Fig. 2). The transverse coordinate v is now replaced by the Cartesian coordinate y, as indicated in Fig. 2. It is assumed that the perturbation is initially confined to the spot area $(|y| \leq s/2)$ and we obtain an estimate of the corresponding maximum strain. This estimate is an upper limit since the disturbance spreads laterally and thereby attenuates. We consider both a "slab" beam (beam fluence profile independent of axial position) and a beam of circular cross-section. The late time harmonic motion associated with a narrow slab beam is discussed in Appendix B.

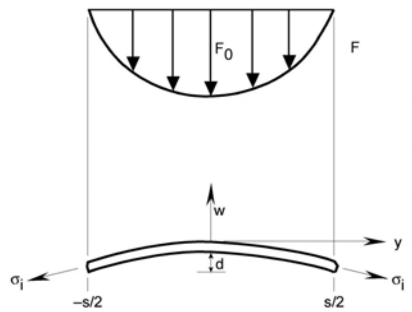

Figure 2. Notation for case of slab laser beam, or circular cross-section beam, incident on slightly curved cylinder section (d/w << 1).

Slab Beam: Using a flat plate approximation, the fundamental frequency and mode shape is of the form

$$\omega = \frac{\pi}{s}\left(\frac{pa}{\rho h}\right)^{1/2}\left[1+\frac{\pi^2}{12}\left(\frac{h}{s}\right)^2\frac{1}{\varepsilon_i}\right]^{1/2} \qquad (21a)$$

$$w \sim \cos\left(\pi y/s\right)\sin\omega t \qquad (21b)$$

Equations 21a and 21b are obtained by introducing $\omega = \bar{\omega}_0$, $w = w_n$, $s = \pi a/n$ and $y = \theta a$ into Eqs. 3a and 12. For convenience, consider an incident fluence profile of the form

$$\frac{F}{F_0} = \cos\left(\frac{\pi y}{s}\right) \qquad (22a)$$

with energy per unit axial length equal to

$$\frac{\bar{J}}{L} \equiv \int_{-s/2}^{s/2} F dy = \frac{2F_0 s}{\pi} \qquad (22b)$$

Here, $\bar{J}$ denotes net beam energy. The vertical velocity at time t = 0 is

$$\frac{\partial w}{\partial t} = (-1)\frac{CF_0}{\rho h}\cos\left(\frac{\pi y}{s}\right) \qquad (23)$$

(Note that Eq. 23 does not apply for values of 2y/s near one, since the magnitude of the fluence in the region is below the threshold needed to induce ablation. However, this region has a small impact on the induced strain. The use of the product CF to characterize the local impulse given to the wall probably requires that s/h be of the order 10, or more.) The induced vibration is then

$$w = (-1)\frac{CF_0}{\rho h\omega}\cos\left(\frac{\pi y}{s}\right)\sin\omega t \qquad (24)$$

The maximum induced strain occurs at y = s/2 and equals

$$\varepsilon_m = \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 = \frac{1}{2}\frac{\left(CF_0\right)^2}{\rho h\ pa} \qquad (25a)$$

For a given cylinder and fluence F_0, the induced maximum strain is independent of slab beam width and is therefore independent of laser beam energy. The corresponding induced stress is

$$\frac{\sigma_m}{\sigma_i} = \frac{1}{2}\left(\frac{CF_0}{pa}\right)^2\frac{E'}{\rho} \qquad (25b)$$

Equations 25a and 25b may be viewed as providing upper bounds. Lateral expansion reduces the vibration amplitude. An estimate of this effect is found as follows. For the case of large n and negligible thickness effects, Eq. 10 has a solution of the form w ~ f(aθ – ct) + g(aθ + ct) which represents waves with velocity $c = \left(\sigma_i/\rho\right)^{1/2}$ traveling in the +θ and –θ directions, respectively. After a quarter period, ωt = π/2, the disturbed area, originally of width s, has expanded to a width of 2s. Equation 25a indicates that, for fixed energy (i.e., sF_0 = constant), the maximum strain is inversely proportional to the square of the beam width. Thus, the maximum strain at ωt = π/2 can be estimated from

$$\varepsilon_m = \frac{1}{8}\frac{\left(CF_0\right)^2}{\rho hpa} \qquad (25c)$$

which is ¼ the value in Eq. 25a. The strain, and corresponding stress, decrease with further expansion of the vibrating region.

Circular cross-section beam: We now consider a circular cross-section beam with radius s/2. Here, the quantity y in Fig. 2 corresponds to a radial coordinate. In order to estimate the initial induced stress, in the present case of cylindrical symmetry, it is necessary to consider the pressure-induced axial stress, as well as the tangential stress σ_i. For unrestrained end walls, the pressure-induced axial stress equals $\sigma_i/2$. Consistent with the objective of estimating an upper limit on the induced stress, we will assume that both the axial and the transverse pressure-induced stresses equal σ_i and neglect thickness effects. In this case the induced vertical perturbations are axisymmetric and are described by the "membrane" vibration equation

$$\frac{\rho}{\sigma_i}\,\frac{\partial^2 w}{\partial t^2} = \frac{1}{y}\,\frac{\partial}{\partial y}\left(y\frac{\partial w}{\partial y}\right) \tag{26}$$

which, for zero displacement at y = s/2, has a solution of the form

$$w \sim J_0\left(2.405\,\frac{2y}{s}\right)\sin\omega t \tag{27a}$$

$$\omega = 2.405\left(\frac{2}{s}\right)\left(\frac{\sigma_i}{\rho}\right)^{1/2} \tag{27b}$$

where $J_m(x)$ is a Bessel Function of the first kind, of order m. We note

$$J_0'(x) = (-1)J_1(x)\ , \quad J_0(2.405) = 0 \tag{28a}$$

$$J_1(1.841) = 0.5819\,(\text{max pt}) \tag{28b}$$

$$\int_0^{2.405} x\,J_0(x)\,dx = 1.2485 \tag{28c}$$

For convenience, we consider an incident beam of the form

$$\frac{F}{F_0} = J_0\left(2.405\,\frac{2y}{s}\right) \tag{29a}$$

which contains a net energy of

$$\overline{J} \equiv 2\pi\int_0^{s/2} F\,y\,dy = 0.4317\,\pi\left(\frac{s}{2}\right)^2 F_0 \tag{29b}$$

The induced vertical velocity at t = 0 is

$$\frac{\partial w}{\partial t} = (-1)\,\frac{CF_0}{\rho h}\,J_0\left(2.405\,\frac{2y}{s}\right) \tag{30}$$

with limitations similar to those observed for Eq. 23. The corresponding vibration and induced strain are

$$w = (-1)\,\frac{CF_0}{\rho h\omega}\,J_0\left(2.405\frac{2y}{s}\right)\sin\omega t \tag{31a}$$

$$\varepsilon = \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 = \frac{1}{2}\left[\frac{CF_0}{\rho h}\sqrt{\frac{\rho}{\sigma_i}}\,J_1\left(2.405\frac{2y}{s}\right)\right]^2 \tag{31b}$$

The maximum strain occurs at 2.405 (2y/s) = 1.841 and equals

$$\varepsilon_m = 0.1693\left[\frac{(CF_0)^2}{\rho h\ pa}\right] \tag{32a}$$

Here again, the maximum strain depends on F_0 and is independent of beam radius and beam energy. The corresponding stress equals

$$\frac{\sigma_m}{\sigma_i} = 0.1693\left[\frac{CF_0}{pa}\right]^2\frac{E'}{\rho} \tag{32b}$$

Equations 32a and 32b represent upper bounds. Lateral expansion decreases the strain. An estimate of this effect, for circular beams, is obtained as follows. Assume that the edge of the disturbance moves with a radial velocity equal to $c = (\sigma_i/\rho)^{1/2}$. After a quarter period, the beam diameter increases from s to s(1 + π/4.810). Equation 32a indicates that for fixed energy (i.e., s^2F_0 = constant), the maximum strain varies inversely as the fourth power of the diameter. Hence the maximum strain at ωt = π/2 can be estimated from

$$\varepsilon_m = \frac{0.1693}{7.469}\,\frac{(CF_0)^2}{\rho h\ pa} \tag{32c}$$

which is a factor 1/7.469 less than the value in Eq. 32a. The strain decreases with further expansion of the vibration area.

Equations 25b and 32b indicate that, for fixed values of the independent variables, the maximum stress induced by a narrow slab beam is a factor 2.95 larger than that induced by a corresponding small diameter circular cross-section beam. This suggests that the use of the narrow slab beam solution to estimate the strain induced by a small diameter circular cross-section beam (as is done in Ref. 4) will lead to an over estimate of induced strain by, approximately, a factor 2.95. The latter is approximate in view of the assumption that the initial axial stress equals the initial tangential stress, σ_i, in the small diameter circular cross-section beam case.

4. Comparison with Sutton

Sutton[2] considers a uniform incident beam with a width of the order of the cylinder diameter. He, in effect, applies flat plate equations (i.e., the equivalent of using Eqs. 12) and thereby neglects curvature effects. The resulting solution can be deduced by taking s = 2a in Appendix B. Sutton excludes the n = 1 mode and claims that the n = 2 mode is the major contributor to cylinder strain. His maximum strain results, in present notation, are (see Eq. B-8)

$$\varepsilon_m = 2\left(\frac{CF}{3\pi}\right)^2 \frac{1}{\rho h\ pa} \tag{33a}$$

$$\frac{\sigma_m}{\sigma_i} = 0.02252\left(\frac{CF}{pa}\right)^2 \frac{E'}{\rho} \tag{33b}$$

We now compare the fluence required to produce a given value of σ_m/σ_i as given by the present model (Eq. 16b) with the value obtained from the Sutton model (Eq. 33b). Let $F_{present}$ denote the fluence in Eq. 16b and let F_{sutton} denote the fluence in Eq. 33b. The ratio of these quantities is

$$\frac{F_{sutton}}{F_{present}} = \left(\frac{1}{0.02252\ \sigma_m/\sigma_i}\right)^{1/2} \tag{34a}$$

$$= 21.07 \qquad \sigma_m/\sigma_i = 0.1 \tag{34b}$$

$$= 12.17 \qquad \sigma_m/\sigma_i = 0.3 \tag{34c}$$

Equations 33 overestimate, by an order of magnitude, the fluence required to achieve a maximum stress of order $\sigma_m/\sigma_i = 0$ (0.1). It should be noted that Eq. 16 refers to the initial transient whereas Eq. 33a refers to the subsequent harmonic motion. However, a comparison of Eq. 33a with Eq. 20b would introduce a factor of only $1/\sqrt{2}$ on the right-hand side of Eq. 34 and thereby would not significantly affect the conclusion. In summary, the major deficiencies of Ref. 23, from the viewpoint of the present study, are (a) curvature effects are neglected and (b) the exclusive use of the quadratic terms in Eq. 8 yields reduced estimates of strain and incorrect scaling laws.

In a more recent study[4], Sutton repeats the above solution for the case s = 2a. He also includes a study of the case s << 2a, for which the flat plate assumption is valid. Sutton claims that, for the case s << 2a, the n = 2 mode is the dominant mode with respect to induced strain. This claim appears to be incorrect for reasons discussed in Appendix B.

5. Cylinder Translation

The cylinder is assumed, initially, to be stationary in space. The laser pulse at t = 0 will induce a uniform translation of the cylinder center of mass. The solution of the present problem, in a coordinate system which moves with the center of mass, is discussed herein.

Consideration of conservation of linear momentum indicates that the laser pulse will induce a cylinder center of mass velocity V, in the θ = 180° direction, equal to

$$\frac{\rho h}{CF} V = \frac{2\theta_s + \sin 2\theta_s}{4\pi} \tag{35a}$$

$$= 1/4 \qquad \theta_s = \pi/2 \tag{35b}$$

$$= \theta_s/\pi \qquad \theta_s \ll 1 \tag{35c}$$

Note that, for $\theta_s = \pi/2$, the magnitude of V is ¼ the value of $\partial w/\partial t$, at θ = 0, in Eq. 13b. Also, V becomes vanishingly small for small width slab beams ($\theta_s \ll 1$).

Equations 2 are applicable in an inertial frame. The initial conditions for Eqs. 2, in a coordinate system fixed with respect to the moving cylinder center of mass, become

$$w = v = 0 \tag{36a}$$

$$\frac{\partial w}{\partial t} = \left(V - \frac{CF}{\rho h}\right)\cos\theta \qquad |\theta| \le \theta_s \tag{36b}$$

$$= V\cos\theta \qquad \theta_s < |\theta| \le \pi \tag{36c}$$

$$\frac{\partial v}{\partial t} = (-1)V\sin\theta \qquad |\theta| \le \pi \tag{36d}$$

Unlike Eqs. 13, Eqs. 36 correspond to zero net linear momentum. The solution of Eqs. 2, with Eqs. 36 as initial conditions, defines the perturbations with respect to the moving cylinder.

Equations 2, with Eqs. 13 as initial conditions, are convenient for finding the perturbations (from the initial cylinder location) at small times. The condition that w and v remain small becomes violated with increase in time. However, the solution of Eqs. 2, with Eqs. 36 as boundary conditions, is valid for all times, $t > 0$, and is convenient for finding the late time solution. The latter is not explored in the present study.

6. Concluding Remarks

The present study provides an upper bound to the maximum hoop stress induced in a thin-walled elastic-pressurized cylinder by a high-energy laser pulse. Further effort is needed to refine these estimates and to establish the longer term vibrational behavior of the cylinder. In addition, end-wall effects may need to be considered in some cases. Finally, the relationship between the present linear stress/strain estimates and structural failure mechanisms, in pressurized cylinders, needs further study.

7. References

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Appendix A: Strain Expressions

Strain is related to the perturbations v, w in the following section. Let $a\Delta\theta$ denote an elementary arc section.

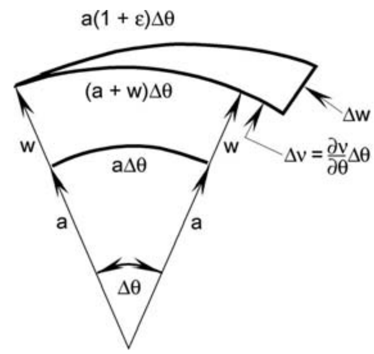

Figure A-1. Notation used for evaluation of strain ε.

The strain induced by a displacement can be expressed (Figure A-1)

$$\left[(1+\varepsilon)\,a\Delta\theta\right]^2 = (\Delta w)^2 + \left[(a+w)\,\Delta\theta + \frac{\partial v}{\partial\theta}\Delta\theta\right]^2 \quad \text{(A-1)}$$

Expansion, retention of leading terms and application of the limit $\Delta\theta \to 0$ indicates

$$\varepsilon = \frac{w}{a} + \frac{\partial v}{a\partial\theta} + \frac{1}{2}\left(\frac{\partial w}{a\partial\theta}\right)^2 + \frac{1}{2}\left(\frac{\partial v}{a\partial\theta}\right)^2 \quad \text{(A-2)}$$

In the case of harmonic motion, the linear terms in Eq. A-2 dominate for moderate values of n and the quadratic terms dominate for large values of n.

A summation is required when there are multiple modes. The strain associated with moderate values of n is (recalling that $nv_n^- / w_n^- = -1$ for this case)

$$\varepsilon = \sum_n \left(1+n^2\right)\left(w_n^+ / a\right) \cos n\theta \sin \omega_n^- t \quad \text{(A-3)}$$

The maximum strain occurs at $\theta = 0$. For large n, where the flat plate approximation applies, the strain is found from

$$\varepsilon = \frac{1}{2}\left[\sum_n \frac{nw_n^-}{a} \sin n\theta \sin \omega_n^- t\right]^2 \quad \text{(A-4)}$$

If there is a dominant mode number n, the maximum strain occurs at $n\theta = \pi/2$.

Equation A-2 defines the tensile strain perturbation associated with the mid section of the cylinder wall. The total strain at the mid section is found from the sum $\varepsilon + \varepsilon_i$. The strain at the outer surface of the cylinder is found by adding the bending strain $\varepsilon_b = h/(2a)$.

Appendix B: Fourier Analysis

For the case of an incident laser beam with small width, the flat plate approximation (Eq. 12) is applicable. The harmonic motion, which follows the initial transient, can then be found by a Fourier analysis. This solution is presented herein. Superscript minus signs are omitted.

Let θ_s denote the value of θ corresponding to the edge of the incident beam, namely,

$$\theta_s = \sin^{-1}\left(s/2a\right) \qquad \text{(B-1)}$$

Define a function f(θ) such that

$$f(\theta) = \cos\theta \qquad |\theta| \le \theta, \qquad \text{(B-2a)}$$

$$= 0 \qquad \theta_s < |\theta| \le \pi \qquad \text{(B-2b)}$$

The function f(θ), expressed as a Fourier series, is

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta \qquad \text{(B-3a)}$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta \qquad \text{(B-3b)}$$

$$= \frac{1}{\pi}\int_{-\theta_s}^{\theta_s} \cos\theta \cos n\theta \, d\theta \qquad \text{(B-3c)}$$

$$= \frac{1}{\pi}\left[\theta_s + \frac{\sin 2\theta_s}{2}\right] \qquad n=1 \qquad \text{(B-3d)}$$

$$= \frac{1}{\pi}\left(\frac{\sin(n-1)\theta_s}{n-1} + \frac{\sin(n+1)\theta_s}{n+1}\right) \qquad n \ne 1 \qquad \text{(B-3e)}$$

The initial conditions, associated with the laser pulse, can be expressed

$$\left(\frac{dw}{dt}\right)_{t=0} = \sum_{n=0}^{\infty} \omega_n w_n \cos n\theta = (-1)\frac{CF}{\rho h} f(\theta) \qquad \text{(B-4)}$$

It follows that

$$(-1)\frac{\rho h}{CF}\omega_n w_n = \frac{a_0}{2} \qquad n = 0 \qquad \text{(B-5a)}$$

$$= a_n \qquad n = 1,2,\ldots \qquad \text{(B-5b)}$$

The flat plate approximation for strain is

$$\varepsilon = \frac{1}{2a^2}\left[\sum_{n=2}^{\infty} n w_n \sin n\theta \sin \omega_n t\right]^2 \qquad \text{(B-6)}$$

Recall

$$\varepsilon_i \omega_o^2 = p/(\rho h a)^2 \qquad \text{(B-7a)}$$

and, for $n^2 >> 1$ and $\left(n^2/12\right)\left(h/a\right)^2 << 1$,

$$\omega_n = n\omega_0\sqrt{\varepsilon_i} \qquad \text{(B-7b)}$$

The strain is then given by

$$\frac{2\rho h p a}{(CF)^2}\varepsilon = \left[\sum_{n=2}^{\infty} a_n \sin n\theta \sin \omega_n t\right]^2 \qquad \text{(B-8)}$$

Equation 33a, in the body of the report, can be obtained by taking n = 2, $\theta_s = \pi/2$, $\theta_2 = \pi/4$ and $\omega_2 t = \pi/2$ in Eq. B-8. We now assume that the beam has a narrow width so that

$$\theta_s = s/(2a) << 1 \qquad \text{(B-9)}$$

It follows that for moderate n,

$$a_n = 2\theta_s/\pi \qquad \text{(B-10)}$$

Let N donate the mode number for which the arc length between radial velocity nodes is equal to s, namely

$$N \equiv \pi a/s = \pi/(2\theta_s) \qquad \text{(B-11)}$$

For values of n such that n/N=O(1),

$$a_n = 2\theta_s / \pi \qquad \text{(B-12)}$$

With further increase in n, the magnitude of a_n approaches zero. Hence we take

$$a_n = 2\theta_s / \pi \qquad n \le N \qquad \text{(B-13)}$$

Substitution into Eq. B-8 yields the following expressions for strain

$$\left(\frac{\pi}{2\theta_s}\right)^2 \frac{3\rho hpa}{(CF)^2} \varepsilon = \left[\sum_{n=2}^{N} \sin n\theta \sin \omega_n t\right]^2 \qquad \text{(B-14a)}$$

$$< N^2 \qquad \text{(B-14b)}$$

The inequality in Eq. B-14b follows from the observation that each term in the summation, on the right hand side of Eq. B-14a, is equal to or less than one. The maximum strain then satisfies the inequality

$$\varepsilon_m < \frac{(CF)^2}{2\rho hpa} \qquad \text{(B-15)}$$

The magnitude and location of the peak strain can be found from a numerical evaluation of the summation in Eq. B-14. Equation B-15 is consistent with Eq. 25a

$$\varepsilon_m = \frac{(CF)^2}{2\rho hpa} \qquad \text{(25a)}$$

which provided an estimate (upper bound) of the peak strain induced during the initial transient.

For moderate values of n, the above solution does not satisfy the initial condition v = 0, as can be seen from Eq. 11b. However, in the limit of small s/(2a), these modes have negligible impact on the solution. The above solution represents an asymptotic limit following the initial transient. The degree to which this solution is achieved depends on the rate of dissipation of the vibrational modes.

Sutton[4] uses a Fourier expansion to describe the oscillations induced by a laser beam for both the s/(2a) = 1 and the s/(2a) << 1 cases. The deficiency of his s/(2a) = 1 solution was discussed in the body of the report (Eq. 34). His treatment of the s/(2/a) << 1 case is now briefly noted. In this case, as in the previous case, he claims that the major contribution to the strain comes from the n=2 mode. The corresponding maximum strain, using the present model, is

$$\varepsilon_{2,m} = \frac{2}{\pi^2} \frac{(CF)^2}{\rho hpa} \theta_s^2 \qquad \text{(B-16)}$$

Equation B-16 differs from Eq. 25a by a factor $\left(2\theta_s/\pi\right)^2$ due to the fact that, after the initial transient, the disturbance is distributed throughout the cylinder. Moreover, the Sutton assumption that the n = 2 mode is the dominant mode, with respect to induced strain, appears to be incorrect in view of Eq. B-14a.

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